Final Exam — Group Theory (WIGT-07)

Thursday April 13, 2017, 09.00h–12.00h

University of Groningen

Instructions

1. Write on each page you hand in your name and student number.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. Your grade for this exam is $E = (P + 10)/10$, where P is the number of points for this exam.

Problem 1 (20 points)

a) Determine all subgroups of S_3. Which of those are normal subgroups?

b) Let $f, g : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R} \setminus \{0, 1\}$ be the functions given by $f(x) := \frac{1}{x}$ and $g(x) := \frac{x-1}{x}$. Consider the group G generated by f and g, where the group operation is defined by composition of functions ($(f \circ g)(x) := f(g(x))$). Show: $G \cong S_3$.

Problem 2 (20 points)

Let $n \in \mathbb{Z}$ and G be a finite group such that $(xy)^n = x^n y^n$ for all $x, y \in G$. We define $G_n := \{x \in G \mid x^n = 1\}$ and $G^n := \{x^n \mid x \in G\}$.

a) Show that G_n and G^n are normal subgroups of G.

b) Show that $|G^n| = [G : G_n]$.

Problem 3 (15 points)

If a group G has a normal subgroup N such that both N and G/N are abelian, we call G *metabelian.*

a) Let $p > q$ be primes such that $q \nmid p - 1$. Show that any group G with $|G| = pq$ is metabelian.

b) Show that every subgroup H of a metabelian group G is also metabelian.

Problem 4 (15 points)

Let G be a finite group and $p \neq q$ prime divisors of $|G|$ such that there exist exactly one Sylow p-group P and one Sylow q-group Q. Show that for any $x \in P$ and $y \in Q$ we have $xyx^{-1}y^{-1} \in P \cap Q$. Conclude that $xy = yx$ and $\mathrm{ord}(xy) = \mathrm{ord}(x)\mathrm{ord}(y)$ for any $x \in P$ and $y \in Q$.

Problem 5 (20 points)

Let $H \subset \mathbb{Z}^4$ be the group generated by the elements $(3, 9, 3, 0)$ and $(4, 2, 0, 2)$. Find the rank and the elementary divisors of $A := \mathbb{Z}^4/H$.

End of test (90 points)

Solutions

Solution 1 (20 points)

a) Determine all subgroups of S_3. Which of those are normal subgroups?
Solution: We have $|S_3| = 6$. Thus, any subgroup besides the trivial ones $\{(1)\}$ and S_3 (2 pts) have order 2 or 3. This implies any non-trivial subgroup is cyclic (2 pts).

This gives that any element of order 2 corresponds to exactly one subgroup of order 2. Thus for any two cycle $(ab) \in S_3$ we have a subgroup of order 2. $(\{(1), (12)\}, \{(1), (13)\}, \{(1), (23)\})$ (2 pts).

In S_3 there are exactly two 3-cycles and any subgroup of order 3 has to include exactly 2 3-cycles. Thus there is exactly one subgroup of order 3 $\{(1), (123), (132)\}$ (2 pts).

Since conjugation fixes cycle types, the only non-trivial normal subgroup of S_3 is the group of order 3 (2 pts).

b) Let $f, g : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R} \setminus \{0, 1\}$ be the functions given by $f(x) := \frac{1}{x}$ and $g(x) := \frac{x-1}{x}$. Consider the group G generated by f and g, where the group operation is defined by composition of functions $((f \circ g)(x) := f(g(x)))$. Show: $G \cong S_3$.
Solution: First possibilty: Compute all possible (there are 6 of them) products of f and g (6 pts). Observe that the group is not commutative (2 pts) and conclude that it is S_3 (2 pts).

Second possibilty: Observe $\text{ord}(f) = 2$ and $\text{ord}(g) = 3$ (4 pts). Compute $f \circ g \circ f = g^{-1}$ (2 pts) and conclude that the group generated by f and g is isomorphic to D_3. By the fact $D_3 \cong S_3$ the assertion follows (2pts).

Third possibilty: The map $f \mapsto (12)$, $g \mapsto (123)$ is a bijective homomorphism. (Here almost the same computations as in the first solution have to be done.)

Solution 2 (20 points)

Let $n \in \mathbb{Z}$ and G be a finite group such that $(xy)^n = x^n y^n$ for all $x, y \in G$. We define $G_n := \{x \in G \mid x^n = 1\}$ and $G^n := \{x^n \mid x \in G\}$.

a) Show that G_n and G^n are normal subgroups of G.
Solution: First we show G^n is a normal subgroup: By the fact $(xy)^n = x^n y^n$ this is straight forward (5 pts). Now consider the homomorphism $\varphi : G \to G^n$ given by $\varphi(x) = x^n$. We have $\ker(\varphi) = G_n$ and thus G_n is a normal subgroup of G. (5 pts)

b) Show that $|G^n| = [G : G_n]$.
Solution: The homomorphism φ in part a) is in fact surjective (2pts) and thus, by the Isomorphism Theorem we have $G/G_n \cong G^n$ (4 pts). This gives $|G/G_n| = |G^n|$ (2 pts). And since $|G/G_n| = [G : G_n]$ the assertion follows (2pts).

Solution 3 (15 points)

If a group G has a normal subgroup N such that both N and G/N are abelian, we call G *metabelian.*

a) Let $p > q$ be primes such that $q \nmid p-1$. Show that any group G with $|G| = pq$ is metabelian.
Solution: By the Sylow Theorem there exists a subgroup P of order p in G (2 pt). Since p is prime, we have that P is abelian (1 pt). Further there exists n_p of those subgroups where $n_p \equiv 1 \pmod{p}$ and $n_p \mid q$ (2 pt). This gives $n_p = 1$ and thus P is normal (2 pt). Thus, the group G/P has q elements and thus is abelian (1 pt). This implies G is metabelian.

b) Show that every subgroup H of a metabelian group G is also metabelian.
Solution: Since G is metabelian we have a normal subgroup N in G that is abelian and G/N is also abelian (1 pt). For any subgroup H of G we consider now the abelian group $H \cap N \subset H$ (2 pts). By the second Isomorphism Theorem we have that $H \cap N$ is normal in H and $H/(H \cap N) \cong HN/N \subset G/N$ (2 pts). Since every subgroup of an abelian group is abelian, we have $H/(H \cap N)$ is abelian (1 pt). Thus, H is metabelian (1 pt).

Solution 4 (15 points)

Let G be a finite group and $p \neq q$ prime divisors of $|G|$ such that there exist exactly one Sylow p-group P and one Sylow q-group Q. Show that for any $x \in P$ and $y \in Q$ we have $xyx^{-1}y^{-1} \in P \cap Q$. Conclude that $xy = yx$ and $\mathrm{ord}(xy) = \mathrm{ord}(x)\mathrm{ord}(y)$ for any $x \in P$ and $y \in Q$.
Solution: Since P and Q have coprime order (2 pts), we have $P \cap Q = \{e\}$ (1 pt). Since P, resp. Q, is the only Sylow p-group, resp q-group, it is normal (2 pts). This gives $xyx^{-1} \in Q$ and $yx^{-1}y^{-1} \in P$ (2 pts). Thus, $xyx^{-1}y^{-1} \in P \cap Q = \{e\}$ (2 pts) and we have $xy = yx$ (2 pts). Further $\gcd(\mathrm{ord}(x), \mathrm{ord}(y)) = 1$ (1 pt) and x, y commute by the last observation (1 pt). Thus, the assertion $\mathrm{ord}(xy) = \mathrm{ord}(x)\mathrm{ord}(y)$ follows (2 pts).

Solution 5 (20 points)

Let $H \subset \mathbb{Z}^4$ be the group generated by the elements $(3, 9, 3, 0)$ and $(4, 2, 0, 2)$. Find the rank and the elementary divisors of $A := \mathbb{Z}^4/H$.
Solution: A straightforward computation gives that the elementry divisors are (6) and the rank is 2.